

AN APPLICATION OF FOURIER ANALYSIS TO RIEMANN SUMS

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ABSTRACT. We develop a method for calculating Riemann sums using Fourier analysis.

1. POISSON SUMMATION FORMULA

Definition 1.1. *If $f \in L^1(\mathcal{R})$, we define;*

$$(f)^\wedge(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy} dx$$

$$(f)_-(y) = f(-y)$$

$$(f)^\vee(y) = \int_{-\infty}^{\infty} f(x)e^{2\pi ixy} dx$$

and, if $g \in L^1([0, 1])$, $m \in \mathcal{Z}$, we define;

$$(g)^\wedge(m) = \int_0^1 g(x)e^{-2\pi ixm} dx$$

Remarks 1.2. *If $f \in \mathcal{S}(\mathcal{R})$, we have that;*

$$f(x) = \int_{-\infty}^{\infty} (f)^\wedge(y)e^{2\pi ixy} dy, \quad (x \in \mathcal{R})$$

and, if $g \in C^\infty([0, 1])$, $(^1)$, the series;

$$\sum_{m \in \mathcal{Z}} (g)^\wedge(m)e^{2\pi ixm}$$

converges uniformly to g on $[0, 1]$. See [4],[2] and [3].

Also observe that $(f)^\vee = (f_-)^\wedge$ and $(f)^\wedge = (f_-)^\vee$.

Theorem 1.3. *Let $f \in \mathcal{S}(\mathcal{R})$, and let;*

¹ By which we mean that $g|_{(0,1)} \in C^\infty(0, 1)$, and there exist $\{g_k \in C[0, 1] : k \in \mathcal{Z}_{\geq 0}\}$, such that $g_k|_{(0,1)} = g^{(k)}$, and $g_k(0) = g_k(1)$.

$$g(y) = \sum_{m \in \mathcal{Z}} f(y + m), \quad (y \in [0, 1])$$

Then $g \in C^\infty([0, 1])$ and the series

$$\sum_{m \in \mathcal{Z}} (f)^\wedge(m) e^{2\pi i y m}$$

converges uniformly to g on $[0, 1]$.

In particular;

$$\sum_{m \in \mathcal{Z}} f(m) = \sum_{m \in \mathcal{Z}} (f)^\wedge(m)$$

Proof. Observe that, as $f \in \mathcal{S}(\mathcal{R})$, for $y_0 \in [0, 1]$, $r \in \mathcal{Z}_{\geq 0}$, $n \geq 2$;

$$\begin{aligned} & \sum_{m \in \mathcal{Z}} \left| \frac{d^r f}{dy^r} \right|_{y_0+m} | \\ & \leq \sum_{m \in \mathcal{Z}} \frac{C_{r,n}}{(1+|y_0+m|^n)} \\ & \leq \sum_{m \in \mathcal{Z}} \frac{C_{r,n}}{(1+|m|^n)} \\ & \leq C_{r,n} + 2C_{r,n} \sum_{m \geq 1} \frac{1}{m^n} \\ & \leq C_{r,n} + 2C_{r,n} (1 + [\frac{y^{-n+1}}{-n+1}]_1^\infty) \\ & = C_{r,n} (1 + 2(1 + \frac{1}{n-1})) \leq 5C_{r,n} \quad (*) \end{aligned}$$

where $C_{r,n} = \sup_{w \in \mathcal{R}} (|w|^n \frac{d^r f}{dx^r} |_w)$

Suppose, inductively, that $\frac{d^r g}{dy^r} |_y = \sum_{m \in \mathcal{Z}} \frac{d^r f}{dy^r} |_y$, for $y_0 \in [0, 1]$,
(²), then, using (*), we have, for $r \geq 1$, that;

$$\frac{d^{r+1} g}{dy^{r+1}} | = \frac{d}{dx} \left(\sum_{m \in \mathcal{Z}} \frac{d^r f_m}{dy^r} \right) = \sum_{m \in \mathcal{Z}} \frac{d^{r+1} f_m}{dy^{r+1}}$$

where $f_m(x) = f(x + m)$, for $m \in \mathcal{Z}$. Moreover, for $r \geq 0$;

$$\frac{d^r g}{dy^r} |_0 = \sum_{m \in \mathcal{Z}} \frac{d^r f_m}{dy^r} |_0 = \sum_{m \in \mathcal{Z}} \frac{d^r f_m}{dy^r} |_1 = \frac{d^r g}{dy^r} |_1$$

It follows that $g \in C^\infty[0, 1]$. Moreover, we have that, for $n \in \mathcal{Z}$;

²Given $\frac{d^r g}{dy^r}$, we interpret $\frac{d^{r+1} g}{dy^{r+1}} |_0 = \lim_{h \rightarrow 0, +} \frac{1}{h} (\frac{d^r g}{dy^r} |_h - \frac{d^r g}{dy^r} |_0)$

$$\begin{aligned}
(g)^\wedge(n) &= \int_0^1 g(y) e^{-2\pi i y n} dx \\
&= \int_0^1 (\sum_{m \in \mathbb{Z}} f(y+m)) e^{-2\pi i y n} dx \\
&= \int_0^1 (\sum_{m \in \mathbb{Z}} f(y+m)) e^{-2\pi i (y+m)n} dx \\
&= \int_{-\infty}^{\infty} f(y) e^{-2\pi i y n} dy = (f)^\wedge(n)
\end{aligned}$$

Using Remark 1.2, the series;

$$\sum_{m \in \mathbb{Z}} \hat{f}(m) e^{2\pi i y m}$$

converges uniformly to g on $[0, 1]$ as required. \square

Lemma 1.4. *If $h \in C^2(\mathcal{R})$, and there exists $C \in \mathcal{R}$, with;*

$$\sup_{x \in \mathcal{R}} (|x|^2 |h(x)|, |x|^2 |h'(x)|, |x|^2 |h''(x)|) \leq C$$

then the Inversion theorem holds for h . That is $(h)^\wedge \in L^1(\mathcal{R})$ and;

$$h(x) = \int_{\mathcal{R}} (h)^\wedge(y) e^{2\pi i x y} dy \quad (x \in \mathcal{R})$$

Proof. The result follows from inspection of the proof in [2], see Remark 0.4. \square

Lemma 1.5. *If h satisfies the conditions of Lemma 1.4, and $f = (h)^\vee$, then $(f)^\wedge = h$.*

Proof. As h satisfies the conditions of Lemma 1.4, so does h_- , and, therefore, the inversion theorem holds for h_- . Then;

$$((h_-)^\wedge)^\vee = (((h_-)^\wedge)^\wedge)_- = (h_-)$$

therefore;

$$(((h_-)^\wedge)^\wedge) = h. \text{ As } f = h^\vee = (h_-)^\wedge, \text{ we have that;}$$

$$(f)^\wedge = (h_-)^\wedge = h$$

\square

Lemma 1.6. *Let f be given by Lemma 1.5. Then, if there exists $D \in \mathcal{R}$, with;*

$$\sup_{x \in \mathcal{R}} (|x|^4 |h(x)|, |x|^4 |h'(x)|, |x|^4 |h''(x)|) \leq D$$

we have that $f \in C^2(\mathcal{R})$, and, moreover, there exists a constant $F \in \mathcal{R}$, such that;

$$\sup_{y \in \mathcal{R}} (|y|^2 |f(y)|, |y|^2 |f'(y)|, |y|^2 |f''(y)|) \leq F.$$

Proof. Letting $E = \|h|_{[-1,1]}\|_{C[-1,1]}$, we have that, for $y \in \mathcal{R}$, $|x| \geq 1$;

$$|h(x)e^{2\pi ixy}| = |h(x)| \leq \frac{D}{|x|^4} \leq \frac{D}{|x|^2}$$

$$|2\pi i x h(x)e^{2\pi ixy}| = 2\pi |x| |h(x)| \leq \frac{2\pi D}{|x|^3} \leq \frac{2\pi D}{|x|^2}$$

$$|-4\pi^2 x^2 h(x)e^{2\pi ixy}| = 4\pi^2 |x|^2 |h(x)| \leq \frac{4\pi^2 D}{|x|^2} \leq \frac{4\pi^2 D}{|x|^2}$$

and, for $y \in \mathcal{R}$, $|x| \leq 1$;

$$|x|^2 |h(x)e^{2\pi ixy}| \leq |h(x)| \leq E$$

$$|x|^2 |2\pi i x h(x)e^{2\pi ixy}| \leq 2\pi |h(x)| \leq 2\pi E$$

$$|x|^2 |-4\pi^2 x^2 h(x)e^{2\pi ixy}| \leq 4\pi^2 |h(x)| \leq 4\pi^2 E$$

Hence;

$$\begin{aligned} & \sup_{x \in \mathcal{R}} \{|x|^2 |h(x)e^{2\pi ixy}|, |x|^2 |2\pi i x h(x)e^{2\pi ixy}|, |x|^2 |-4\pi^2 x^2 h(x)e^{2\pi ixy}|\} \\ & \leq 4\pi^2 \max(D, E) \end{aligned}$$

and $\{h(x)e^{2\pi ixy}, 2\pi i x h(x)e^{2\pi ixy}, -4\pi^2 x^2 h(x)e^{2\pi ixy}\} \subset C(\mathcal{R})$. It follows that, for $y_0 \in \mathcal{R}$, we can differentiate under the integral sign, to obtain that $\{f(y_0), f'(y_0), f''(y_0)\}$ are all defined. By the DCT, using the fact that $-4\pi^2 x^2 h(x) \in L^1(\mathcal{R})$, we obtain that $f'' \in C(\mathcal{R})$, hence, $f \in C^2(\mathcal{R})$. Differentiating by parts, using the fact that;

$$\{h, h', h'', xh, xh', xh'', x^2h, x^2h', x^2h''\} \subset (L^1(\mathcal{R}) \cap C_0(\mathcal{R}))$$

by the hypotheses of Lemma 1.4 and this Lemma, we have that;

$$\begin{aligned}
(h'')^\vee &= -4\pi y^2(h)^\vee = -4\pi y^2 f \\
(4\pi i h' + 2\pi i x h'')^\vee &= ((2\pi i x h'')^\vee)^\vee = -4\pi y^2(2\pi i x h)^\vee = -4\pi y^2 f' \\
(8\pi^2 h + 16\pi^2 x h' + 4\pi^2 x^2 h'')^\vee &= ((4\pi^2 x^2 h'')^\vee)^\vee \\
&= -4\pi y^2(4\pi^2 x^2 h)^\vee = -4\pi y^2 f'', (*)
\end{aligned}$$

We have, by (*), for $|y| \geq 1$, that;

$$\begin{aligned}
|f(y)| &\leq \frac{|(h'')^\vee(y)|}{4\pi y^2} \leq \frac{\|h''\|_{L^1(\mathcal{R})}}{4\pi y^2} \leq \frac{\frac{2D}{3} + 2E''}{4\pi y^2} \\
|f'(y)| &\leq \frac{|(4\pi i h' + 2\pi i x h'')^\vee(y)|}{4\pi y^2} \\
&\leq \frac{\|(4\pi i h' + 2\pi i x h'')\|_{L^1(\mathcal{R})}}{4\pi y^2} \\
&\leq \frac{2\|h'\|_{L^1(\mathcal{R})} + \|x h''\|_{L^1(\mathcal{R})}}{2y^2} \\
&\leq \frac{2(\frac{2D}{3} + 2E') + D + 2E''}{2y^2} \\
|f''(y)| &\leq \frac{|(8\pi^2 h + 16\pi^2 x h' + 4\pi^2 x^2 h'')^\vee(y)|}{4\pi y^2} \\
&\leq \frac{\|(8\pi^2 h + 16\pi^2 x h' + 4\pi^2 x^2 h'')\|_{L^1(\mathcal{R})}}{4\pi y^2} \\
&\leq \frac{2\pi\|h\|_{L^1(\mathcal{R})} + 4\pi\|x h'\|_{L^1(\mathcal{R})} + \pi\|x^2 h''\|_{L^1(\mathcal{R})}}{y^2} \\
&\leq \frac{2\pi(2\frac{D}{3} + 2E) + 4\pi(D + 2E') + \pi(2D + 2E'')}{y^2}
\end{aligned}$$

where $E' = \|h'\|_{[-1,1]} \|C_{[-1,1]}$ and $E'' = \|h''\|_{[-1,1]} \|C_{[-1,1]}$

For $|y| \leq 1$, we have that;

$$\begin{aligned}
|f(y)| &\leq \|h\|_{L^1(\mathcal{R})} \leq \frac{2D}{3} + E \\
|f'(y)| &\leq \|(2\pi i x h)\|_{L^1(\mathcal{R})} \leq 2\pi D + 2E \\
|f''(y)| &\leq \|-4\pi^2 x^2 h\|_{L^1(\mathcal{R})} \leq 8\pi^2 D + 2E
\end{aligned}$$

Hence, we can take $F = \max(8\pi^2 D + 2E, \frac{22\pi D}{3} + 4\pi E + 8\pi E' + 2\pi E'')$

□

Definition 1.7. Let f be given by satisfying the conditions of Lemmas 1.5 and 1.6, we let;

$$g(y) = \sum_{m \in \mathcal{Z}} f(y + m), \quad (y \in [0, 1])$$

Lemma 1.8. Let g be given by Definition 1.7, then $g \in C^2[0, 1]$.

Proof. Using Lemma 1.6 and Weierstrass' M-test, we have that the series;

$$\sum_{m \in \mathcal{Z}} f(y + m), \sum_{m \in \mathcal{Z}} f'(y + m), \sum_{m \in \mathcal{Z}} f''(y + m)$$

are uniformly convergent on $[0, 1]$. It follows, that $g \in C^2(0, 1)$, and clearly;

$$g'_+(0) = \sum_{m \in \mathcal{Z}} f'(m) = \sum_{m \in \mathcal{Z}} f'(m + 1) = g'_-(1)$$

hence, $g \in C^2[0, 1]$.

□

Lemma 1.9. Let $f \in L^1(\mathcal{R})$, such that;

$$g(y) = \sum_{m \in \mathcal{Z}} f(y + m)$$

is defined, for $y \in [0, 1]$. Then, if $g \in C^2[0, 1]$, we have that the series $\sum_{m \in \mathcal{Z}} (f)^\wedge(m) e^{2\pi i y m}$ converges uniformly to g on $[0, 1]$. In particular;

$$\sum_{m \in \mathcal{Z}} f(m) = \sum_{m \in \mathcal{Z}} (f)^\wedge(m)$$

Proof. Following through the calculation in Theorem 1.3, we have that $g \in L^1([0, 1])$, and $(g)^\wedge(m) = (f)^\wedge(m)$, for $m \in \mathcal{Z}$. Using the result of [3] or [4], we obtain the second part, the final claim is clear.

□

Lemma 1.10. Let f be given by satisfying the conditions of Lemmas 1.5 and 1.6, with respect to h , then;

$$\sum_{m \in \mathcal{Z}} f(m) = \sum_{m \in \mathcal{Z}} h(m)$$

Proof. Using Lemmas 1.5 and 1.6, we have that $g \in C^2[0, 1]$, where g is defined by 1.8, and $(f)^\wedge(m) = h(m)$, for $m \in \mathcal{Z}$. By Lemmas 1.8

and 1.9, we have that;

$$\sum_{m \in \mathbb{Z}} f(m) = \sum_{m \in \mathbb{Z}} (f)^\wedge(m)$$

Hence;

$$\sum_{m \in \mathbb{Z}} f(m) = \sum_{m \in \mathbb{Z}} h(m)$$

as required. □

Lemma 1.11. *If $s \in \mathbb{Z}_{\geq 2}$, s even, then;*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{(-1)^{\frac{s+2}{2}} (2\pi)^s B_s}{2(s!)}$$

Proof. The proof of this result can be found in [4]. □

Definition 1.12. *If $s \in \mathbb{C}$, with $\operatorname{Re}(s) \geq 4$, and $r \in \mathbb{Z}_{\geq 1}$, we define;*

$$h_{s,r}(x) = \frac{1}{x^s}, \quad (x \geq r)$$

$$h_{s,r}(x) = \frac{(-1)^s}{x^s} = \frac{e^{-i\pi s}}{x^s}, \quad (x \leq -r)$$

Remarks 1.13. $h_{s,r}$ is symmetric, that is $h_{s,r}(x) = h_{s,r}(-x)$, for $|x| \geq r$.

Lemma 1.14. *There exists a polynomial $p_{s,r}$ of degree $2r + 3$, with the properties;*

(i). $p_{s,r}$ is symmetric, that is $p_{s,r}(x) = p_{s,r}(-x)$, for $x \in \mathbb{R}$.

(ii). $p_{s,r}(n) = \frac{1}{n^s}$, for $1 \leq n \leq r$.

(iii). $p_{s,r}^{(k)}(r) = h_{s,r}^{(k),+}(r)$, ($0 \leq k \leq 2$)

(iv). $p_{s,r}^{(k)}(-r) = h_{s,r}^{(k),-}(-r)$, ($0 \leq k \leq 2$)

Proof. We let, for $1 \leq j \leq 1 + r$, $1 \leq k \leq r$;

$$\overline{A}_r = \begin{pmatrix} 1 & 1 & \dots & 1 & \dots & 1 \\ 1 & 2^2 & \dots & 2^{2j} & \dots & 2^{2(r+1)} \\ \dots & & & & & \\ 1 & k^2 & \dots & k^{2j} & \dots & k^{2(r+1)} \\ \dots & & & & & \\ 1 & r^2 & \dots & r^{2j} & \dots & r^{2(r+1)} \\ 0 & 2r & \dots & 2jr^{2j-1} & \dots & 2(r+1)r^{2r+1} \\ 0 & 2 & \dots & 2j(2j-1)r^{2j-2} & \dots & 2(r+1)(2r+1)r^{2r} \end{pmatrix}$$

$$\overline{b}_{s,r} = \begin{pmatrix} 1^{-s} \\ 2^{-s} \\ \dots \\ k^{-s} \\ \dots \\ r^{-s} \\ -sr^{-(1+s)} \\ s(s+1)r^{-(2+s)} \end{pmatrix}$$

We have that $\det(\overline{A}_r) \neq 0$, hence, we can solve the equation $\overline{A}_r(\overline{a}_{s,r}) = \overline{b}_{s,r}$. Let $p_{s,r}(x) = \sum_{j=0}^{r+1} (\overline{a}_{s,r})_{(j+1)} x^{2j}$. We have, by construction, that $p_{s,r}(-x) = p_{s,r}(x)$, and $p_{s,r}^{(k)}(r) = h_{s,r}^{(k),+}(r)$. As both $p_{s,r}$ and $h_{s,r}$ are symmetric, we also have that, $p_{s,r}^{(k)}(r) = h_{s,r}^{(k),-}(-r)$, as required. \square

Definition 1.15. *We define;*

$$g_{s,r}(x) = h_{s,r}(x), \text{ (if } |x| \geq r \text{)}$$

$$g_{s,r}(x) = p_{s,r}(x), \text{ (if } |x| \leq r \text{)}$$

Lemma 1.16. *We have that $g_{s,r} \in C^2(\mathcal{R})$, $g_{s,r}$ is symmetric, and, moreover, the hypotheses of Lemmas 1.4 and 1.6 hold for $g_{s,r}$.*

Proof. The fact that $g_{s,r} \in C^2(\mathcal{R})$ follows immediately from Conditions (iii) and (iv) of Lemma 1.14. The symmetry condition is a consequence of Condition (i). If $x \geq r$, we have that;

$$|g_{s,r}(x)| \leq |x^{-Re(s)}| |x^{-Im(s)}| \leq |x|^{-4}$$

Hence, as $g_{s,r}$ is symmetric, $|g_{s,r}(x)| \leq |x|^{-4}$, for $|x| \geq r$.

If $|x| \leq r$;

$$|g_{s,r}(x)| = |p_{s,r}(x)| \leq r^{2r+2} \sum_{j=0}^{r+1} |(\bar{a}_{s,r})_{(j+1)}| \leq r^{2r+2} \sqrt{r+2} \|\bar{a}_{s,r}\|$$

It follows that $\sup_{x \in \mathcal{R}} (|x|^4 |g_{s,r}(x)|) \leq \max(1, r^{2r+6} \sqrt{r+2} \|\bar{a}_{s,r}\|)$. Similarly, as $g'_{s,r}(x) = \frac{-s}{x^{s+1}}$, $g''_s(x) = \frac{s(s+1)}{x^{s+2}}$, $|x| > r$, then, if $|x| > r$, we have that;

$$|g'_{s,r}(x)| \leq |s| |x|^{-5}$$

$$|g''_{s,r}(x)| \leq |s| |s-1| |x|^{-6}$$

and, if $|x| \leq r$;

$$|g'_{s,r}(x)| = |p'_{s,r}(x)| \leq r^{2r+2} (\sum_{j=1}^{r+1} |2j(\bar{a}_{s,r})_{(j+1)}|) \leq 2(r+1)r^{2r+2} \sqrt{r+2} \|\bar{a}_{s,r}\|$$

$$|g''_s(x)| = |p''_s(x)| \leq r^{2r+2} (\sum_{j=1}^{r+1} |2j(2j-1)(\bar{a}_{s,r})_{(j+1)}|) \leq (2r+2)(2r+1)r^{2r+2} \sqrt{r+2} \|\bar{a}_s\|$$

so that;

$$\sup_{x \in \mathcal{R}} (|x|^5 |g'_{s,r}(x)|) \leq \max(|s|, 2(r+1)r^{2r+7} \sqrt{r+2} \|\bar{a}_s\|)$$

$$\sup_{x \in \mathcal{R}} (|x|^6 |g''_{s,r}(x)|) \leq \max(|s| |s-1|, (2r+2)(2r+1)r^{2r+8} \sqrt{r+2} \|\bar{a}_s\|)$$

(*)

It follows that Lemmas 1.4 and 1.6 holds for $g_{s,r}$, with $C = D = \max(|s| |s-1|, (2r+2)(2r+1)r^{2r+8} \sqrt{r+2} \|\bar{a}_s\|)$.

□

Definition 1.17. We let $f_{s,r}(y) = \int_{\mathcal{R}} g_{s,r}(x) e^{2\pi i x y} dx$.

$$R_{s,r,1} = \sum_{n \in \mathbb{Z}_{\neq 0}} \left(\int_r^\infty \frac{e^{2\pi i n x}}{x^s} dx \right)$$

$$R_{s,r,2} = \sum_{n \in \mathbb{Z}_{\neq 0}} \left(\int_r^\infty \frac{e^{-2\pi i n x}}{x^s} dx \right)$$

$$R_{s,r} = R_{s,r,1} + R_{s,r,2}$$

$$P_{s,r,1} = \sum_{n \in \mathbb{Z}_{\neq 0}} \left(\int_0^r p_{s,r}(x) e^{2\pi i n x} dx \right)$$

$$P_{s,r,2} = \sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\int_0^r p_{s,r}(x) e^{-2\pi i n x} dx \right)$$

$$P_{s,r} = P_{s,r,1} + P_{s,r,2}$$

Lemma 1.18. *We have that $f_{s,r}$ is symmetric, and $f_{s,r}$ satisfies the conclusions of Lemmas 1.5 and 1.6. Moreover;*

$$f_{s,r}(0) + P_{s,r} + R_{s,r} = p_{s,r}(0) + 2 \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Proof. The second part follows immediately from Definition 1.17, and Lemmas 1.16, 1.5 and 1.6. It follows that $f_{s,r} \in L^1(\mathcal{R})$, and;

$$\begin{aligned} f_{s,r}(-y) &= \int_{\mathcal{R}} g_{s,r}(x) e^{-2\pi i x y} dx \\ &= \int_{\mathcal{R}} g_{s,r}(-x) e^{2\pi i x y} dx \\ &= \int_{\mathcal{R}} g_{s,r}(x) e^{2\pi i x y} dx \\ &= f_{s,r}(y) \end{aligned}$$

Hence, $f_{s,r}$ is symmetric. By Lemma 1.10, we have that;

$$\sum_{n \in \mathbb{Z}} f_{s,r}(n) = \sum_{n \in \mathbb{Z}} g_{s,r}(n)$$

As both $f_{s,r}$ and $g_{s,r}$ are symmetric, using Definition 1.15 and property (ii) of Lemma 1.14, we obtain;

$$\begin{aligned} &f_{s,r}(0) + P_{s,r} + R_{s,r} \\ &= f_{s,r}(0) + \sum_{n \in \mathbb{Z} \setminus \{0\}} f_{s,r}(n) \\ &= g_{s,r}(0) + 2 \left(\sum_{n=1}^{\infty} g_{s,r}(n) \right) \\ &= p_{s,r}(0) + 2 \left(\sum_{n=1}^{\infty} \frac{1}{n^s} \right) \end{aligned}$$

□

Lemma 1.19. *We have that;*

$$|R_{s,r}| \leq \frac{2|s|^2}{3(Re(s)+1)r^{Re(s)+1}}$$

$$P_{s,r} = p_{s,r}(0) + p_{s,r}(r) + 2 \sum_{l=1}^{r-1} p_{s,r}(l) - 2 \int_0^r p_{s,r}(x) dx$$

Proof. We have that;

$$\begin{aligned} R_{s,r,1} &= \sum_{n \in \mathcal{Z}_{\neq 0}} \frac{-1}{2\pi i n r^s} + \sum_{n \in \mathcal{Z}_{\neq 0}} \frac{s}{2\pi i n} \int_r^\infty \frac{e^{2\pi i n x}}{x^{s+1}} dx \\ &= \sum_{n \in \mathcal{Z}_{\neq 0}} \frac{-s}{r^{s+1}(2\pi i n)^2} + \sum_{n \in \mathcal{Z}_{\neq 0}} \frac{s(s+1)}{(2\pi i n)^2} \int_r^\infty \frac{e^{2\pi i n x}}{x^{s+2}} dx \\ &= \frac{2s}{4r^{s+1}\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} + D_{s,r,1} \\ &= \frac{s}{2r^{s+1}\pi^2} \frac{\pi^2}{6} + D_{s,r,1} \\ &= \frac{s}{12r^{s+1}} + D_{s,r,1} \end{aligned}$$

where;

$$|D_{s,r,1}| \leq \frac{|s(s+1)|C_{s,r,1}}{4\pi^2} \sum_{n \in \mathcal{Z}_{\neq 0}} \frac{1}{n^2} = \frac{|s(s+1)|C_{s,r,1}}{12}$$

$$\text{and } C_{s,r,1} \leq \int_r^\infty \frac{1}{|x^{s+2}|} dx$$

$$= \int_r^\infty \frac{dx}{x^{Re(s)+2}}$$

$$= \frac{1}{(Re(s)+1)r^{Re(s)+1}}$$

It follows that;

$$\begin{aligned} |R_{s,r,1}| &\leq \frac{|s|}{12r^{Re(s)+1}} + \frac{|s(s+1)|}{12(Re(s)+1)r^{Re(s)+1}} \\ &\leq \frac{|s|(Re(s)+1)}{12(Re(s)+1)r^{Re(s)+1}} + \frac{|s(s+1)|}{12(Re(s)+1)r^{Re(s)+1}} \\ &= \frac{|s|(Re(s)+1)+|s(s+1)|}{12(Re(s)+1)r^{Re(s)+1}} \\ &\leq \frac{2|s(s+1)|}{12(Re(s)+1)r^{Re(s)+1}} \\ &\leq \frac{|s|^2}{3(Re(s)+1)r^{Re(s)+1}} \end{aligned}$$

$$\text{Similarly, } |R_{s,r,2}| \leq \frac{|s|^2}{3(Re(s)+1)r^{Re(s)+1}}, \text{ so that } |R_{s,r}| \leq \frac{2|s|^2}{3(Re(s)+1)r^{Re(s)+1}}.$$

We have that;

$$\begin{aligned}
P_{s,r,1} &= \sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\int_0^r p_{s,r}(x) e^{2\pi i n x} dx \right) \\
&= \sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\sum_{l=0}^{r-1} \int_l^{l+1} p_{s,r}(x) e^{2\pi i n x} dx \right) \\
&= \sum_{l=0}^{r-1} \left(\sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\int_l^{l+1} p_{s,r}(x) e^{2\pi i n x} dx \right) \right) \\
&= \sum_{l=0}^{r-1} \left(\sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\int_0^1 p_{s,r}(x+l) e^{2\pi i n(x+l)} dx \right) \right) \\
&= \sum_{l=0}^{r-1} \left(\sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\int_0^1 p_{s,r}(x+l) e^{2\pi i n x} dx \right) \right) \\
&= \sum_{l=0}^{r-1} \left(\sum_{n \in \mathbb{Z} \setminus \{0\}} (p_{s,r}^l)^\vee(n) \right) \\
&= \sum_{l=0}^{r-1} \left(\sum_{n \in \mathbb{Z}} (p_{s,r}^l)^\vee(n) - \int_0^1 p_{s,r}^l dx \right), \quad (3)
\end{aligned}$$

³If $f \in (C[0,1] \cap C^2(0,1))$, and there exist $\{a_{+,j}, a_{-,j} : 0 \leq j \leq 2\} \subset \mathcal{C}$, with $\lim_{x \rightarrow 0^+} f^{(j)}(x) = a_{+,j}$ and $\lim_{x \rightarrow 1^-} f^{(j)}(x) = a_{-,j}$, (\dagger), for $0 \leq j \leq 2$, then a classical result in the theory of Fourier series, says that;

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N (f)^{(\wedge)}(n) e^{2\pi i n x} = f(x) \quad (x \in (0,1))$$

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N (f)^{(\wedge)}(n) = \frac{a_{+,0} + a_{-,0}}{2}$$

We give a simple proof of this result. First observe that there exists a polynomial $p \in \mathcal{C}[x]$, with $\deg(p) = 5$, such that $p^{(j)}(0) = 0$ and $p^{(j)}(1) = a_{-,j} - a_{+,j}$, for $0 \leq j \leq 2$. This follows from the fact that we can find $\bar{c} \in \mathcal{C}^3$, such that $\overline{M} \cdot \bar{c} = \bar{a}$, where $\bar{a}(j) = a_{-,j-1} - a_{+,j-1}$, for $1 \leq j \leq 3$, and;

$$\overline{M} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 12 & 20 \end{pmatrix}$$

as $\det(\overline{M}) \neq 0$, and, setting $p(x) = \sum_{k=0}^2 c_k x^{3+k}$. We have that $p + f \in C^2(S^1)$, in which case the result follows from [3]. Hence, it is sufficient to verify the result for the powers $\{x^k : 0 \leq k \leq 5\}$. We have that, for $k \geq 1$, $n \in \mathbb{Z} \setminus \{0\}$;

$$\begin{aligned}
&\int_0^1 x^k e^{-2\pi i n x} dx \\
&= \left[\frac{x^k e^{-2\pi i n x}}{-2\pi i n} \right]_0^1 + \frac{k}{2\pi i n} \int_0^1 x^{k-1} e^{-2\pi i n x} dx \\
&= \frac{-1}{2\pi i n} + \frac{k}{2\pi i n} \int_0^1 x^{k-1} e^{-2\pi i n x} dx \\
&= \int_0^1 x^k e^{-2\pi i n x} dx \\
&= - \left(\sum_{l=1}^k \frac{k!}{(k-l+1)!(2\pi i n)^l} \right) + \int_0^1 e^{-2\pi i n x} dx \\
&= - \left(\sum_{l=1}^k \frac{k!}{(k-l+1)!(2\pi i n)^l} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{r-1} \left(\frac{p_{s,r}^l(1) + p_{s,r}^l(0)}{2} - \int_0^1 p_{s,r}^l dx \right) \\
&= \sum_{l=0}^{r-1} \left(\frac{p_{s,r}(l+1) + p_{s,r}(l)}{2} - \int_0^1 p_{s,r}(x+l) dx \right) \\
&= \sum_{l=0}^{r-1} \left(\frac{p_{s,r}(l+1) + p_{s,r}(l)}{2} - \int_l^{l+1} p_{s,r}(x) dx \right) \\
&= \frac{p_{s,r}(0) + p_{s,r}(r)}{2} + \sum_{l=1}^{r-1} p_{s,r}(l) - \int_0^r p_{s,r}(x) dx
\end{aligned}$$

Similarly;

$$P_{s,r,2} = \frac{p_{s,r}(0) + p_{s,r}(r)}{2} + \sum_{l=1}^{r-1} p_{s,r}(l) - \int_0^r p_{s,r}(x) dx$$

so that;

$$P_{s,r} = P_{s,r,1} + P_{s,r,2}$$

$$\begin{aligned}
&\lim_{N \rightarrow \infty} \sum_{n=-N}^N (x^k)^\wedge(n) \\
&= \frac{1}{k+1} - 2 \sum_{l=1}^k \sum_{n=1}^{\infty} \frac{k!}{(k-l+1)!(2\pi i n)^l} \\
&\text{Case } k=1, \text{ we obtain } S_k = \frac{1}{2} \\
&k=2, S_k = \frac{1}{3} + \frac{2.2}{4\pi^2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \\
&k=3, S_k = \frac{1}{4} + \frac{2.3}{4\pi^2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \\
&k=4, S_k = \frac{1}{5} + \frac{2.4}{4\pi^2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) - \frac{2.24}{16\pi^4} \left(\sum_{n=1}^{\infty} \frac{1}{n^4} \right) \\
&k=5, S_k = \frac{1}{6} + \frac{2.5}{4\pi^2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) - \frac{2.120}{16\pi^4} \left(\sum_{n=1}^{\infty} \frac{1}{n^4} \right)
\end{aligned}$$

Using Lemma 1.11, we have that;

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{-\pi [\cot(\pi z)z]^{(2)}|_0}{2.2!} = \frac{-\pi. -4\pi}{6.2.2!} = \frac{\pi^2}{6} \\
\sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{-\pi [\cot(\pi z)z]^{(4)}|_0}{2.4!} = \frac{-\pi. -48\pi^3}{90.2.4!} = \frac{\pi^4}{90} \\
S_2 &= \frac{1}{3} + \frac{2.2}{4\pi^2} \left(\frac{\pi^2}{6} \right) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \\
S_3 &= \frac{1}{4} + \frac{2.3}{4\pi^2} \frac{\pi^2}{6} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\
S_4 &= \frac{1}{5} + \frac{2.4}{4\pi^2} \frac{\pi^2}{6} - \frac{2.24}{16\pi^4} \frac{\pi^4}{90} = \frac{1}{2} \\
S_5 &= \frac{1}{6} + \frac{2.5}{4\pi^2} \frac{\pi^2}{6} - \frac{2.60}{16\pi^4} \frac{\pi^4}{90} = \frac{1}{2}
\end{aligned}$$

$$= p_{s,r}(0) + p_{s,r}(r) + 2 \sum_{l=1}^{r-1} p_{s,r}(l) - 2 \int_0^r p_{s,r}(x) dx$$

□

Lemma 1.20. *If $\operatorname{Re}(s) \geq 4$, $r \geq 1$, we have that;*

$$\begin{aligned} & \sum_{n=r}^{\infty} \frac{1}{n^s} \\ &= \int_r^{\infty} \frac{dx}{x^s} + \frac{p_{s,r}(r)}{2} + \frac{R_{s,r}}{2} \\ &= \frac{1}{(s-1)r^{s-1}} + \frac{R_{s,r}}{2} + \frac{r^s}{2} \\ & \text{If } r \geq 2; \\ & \sum_{n=1}^{r-1} \frac{1}{n^s} \\ &= \sum_{j=0}^{r+1} (\bar{a}_{s,r})_{j+1} \left(\frac{B_{2j+1}(r)}{2j+1} \right) \end{aligned}$$

Proof. The first claim is just a simple rearrangement of the claim in Lemma 1.18, using Lemma 1.19. We have that;

$$\int_r^{\infty} \frac{dx}{x^s} = \frac{1}{(s-1)r^{s-1}}$$

and $\frac{p_{s,r}(r)}{2} = \frac{r^s}{2}$, by property (ii) in Lemma 1.14. Moreover;

$$\begin{aligned} & \sum_{n=1}^{r-1} \frac{1}{n^s} \\ &= \sum_{l=1}^{r-1} p_{s,r}(l) \\ &= \sum_{l=0}^{r-1} \sum_{j=0}^{r+1} (\bar{a}_{s,r})_{j+1} l^{2j} \\ &= \sum_{j=0}^{r+1} (\bar{a}_{s,r})_{j+1} \left(\sum_{l=0}^{r-1} l^{2j} \right) \\ &= \sum_{j=0}^{r+1} (\bar{a}_{s,r})_{j+1} \left(\frac{B_{2j+1}(r) - B_{2j+1}(0)}{2j+1} \right) \\ &= \sum_{j=0}^{r+1} (\bar{a}_{s,r})_{j+1} \left(\frac{B_{2j+1}(r)}{2j+1} \right) \end{aligned}$$

□

Remarks 1.21. *Using Lemma 1.19, we have that $\lim_{r \rightarrow \infty} |R_{s,r}| = 0$, hence, Lemma 1.20 reduces the calculation of $\sum_{n=1}^{\infty} \frac{1}{n^s}$ to a calculation*

involving Bernoulli polynomials. Moreover, letting $A_{s,r} = \sum_{n=r}^{\infty} \frac{1}{n^s}$, we have that;

$$\int_r^{\infty} \frac{dx}{|x^s|} \leq |A_{s,r}| \leq \int_{r-1}^{\infty} \frac{dx}{|x^s|}$$

$$\int_r^{\infty} \frac{dx}{x^{\operatorname{Re}(s)}} \leq |A_{s,r}| \leq \int_{r-1}^{\infty} \frac{dx}{|x^s|}$$

$$\frac{1}{(\operatorname{Re}(s)-1)r^{\operatorname{Re}(s)-1}} \leq |A_{s,r}| \leq \frac{1}{3(r-1)^3}$$

Observing that;

$$\frac{|s|^2}{3(\operatorname{Re}(s)+1)r^{\operatorname{Re}(s)+1}} \leq \frac{1}{(\operatorname{Re}(s)-1)r^{\operatorname{Re}(s)-1}}$$

if $r \geq |s| \sqrt{\frac{(\operatorname{Re}(s)-1)}{3(\operatorname{Re}(s)+1)}}$, we have that the estimate $\sum_{j=0}^{r+1} (\bar{a}_{s,r})_{j+1} (\frac{B_{2j+1}(r)}{2j+1}) + \frac{1}{(s-1)r^{s-1}} + \frac{1}{2r^s}$ improves upon $\sum_{j=0}^{r+1} (\bar{a}_{s,r})_{j+1} (\frac{B_{2j+1}(r)}{2j+1})$, for sufficiently large values of r . The coefficients $(\bar{a}_{s,r})_j$, $1 \leq j \leq r+2$ can be computed using simple linear algebra. The computation of absolutely convergent Riemann sums, and their differences, occurs in the evaluation of $\zeta(s)$, for $0 < \operatorname{Re}(s) < 1$, it is well known that $\zeta(s) \neq 0$, for $\operatorname{Re}(s) \geq 1$. It is hoped that the above method might lead to some progress in the direction of solving the famous Riemann hypothesis, that, $\zeta(s) = 0$ iff $\operatorname{Re}(s) = \frac{1}{2}$ or $s = -2w$, for $w \in \mathbb{Z}_{\geq 1}$, see [1].

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